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AN INVARIANT OF FREE CENTRAL EXTENSIONS

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In memory of my father Israel Rosset

If G is a free abelian finitely generated group, the 'most-general' 2-cocycle on G with trivial action is the map $G \times G \xrightarrow{f} \Lambda^2 G$, defined as follows. Let $e_1, \dots, e_n$ be a basis of G and f the bilinear map satisfying $f(e_i, e_j) = e_i \wedge e_j$ if $i < j$ and $=0$ if $i \geq j$. We show that $K^\alpha G$ has global dimension 1 where K is the field of fractions of the group ring $\mathbb{C}[\Lambda^2 G]$ and $\alpha \in H^2(G, K^*)$ is represented by the map above. More generally for every finitely generated group we define an invariant $\xi(G)$ and $\mathrm{gl.dim}(K^\alpha G) = 1$ is equivalent to $\xi(G) = 1$ for G free abelian. We also show that if $G_1, \dots, G_n$ are non-commutative free groups, then $\xi(G_1 \times \cdots \times G_n) = n$. In general, if G is not a torsion group, $1 \leq \xi(G) \leq \mathrm{cd}_{\mathbb{C}}(G)$.

0. Introduction

A presentation of a group G is a surjective homomorphism $\pi : F \to G$ where F is a free group. To a presentation can be associated a central extension with torsion free kernel as follows. Let $R = \ker(\pi)$ (the subgroup of 'relations'). The extension

$$1 \to R/[F, R] \to F/[F, R] \to G \to 1 ,$$

where the map $F/[F, R] \to G$ is that induced by π, is obviously central. (Notation: $[x, y] = xyx^{-1}y^{-1}$ and $[A, B] =$ subgroup generated by the set $\{[a, b]: a \in A, b \in B\}$.) Since $R/[F, R]$ is central its torsion, $\mathcal{T}$, is a normal subgroup of $F/[F, R]$ and we divide it out. Let $R/[F, R]/\mathcal{T} = A$. It is a central torsion free subgroup of $\Gamma = F/[F, R]/\mathcal{T}$. The extension

$$1 \to A \to \Gamma \to G \to 1$$

is what we call the associated *free* (torsion free) central extension (of G).

As an example suppose G is free abelian of rank m. Then, taking F to be free on m generators, A is free on $\frac{1}{2}m(m-1)$ generators and can be identified with $G \wedge G$ (see Section 4).

Although there is nothing canonical about presentations and their associated central extensions one knows that very interesting invariants, $H_2(G, \mathbb{Z})$ not the least, can be extracted from a presentation. In this paper we exhibit a numerical invariant of G which is defined by the associated free torsion free central extension of G, and show by some examples that it is not a trivial invariant.

If $\mathbb{C}$ is the complex field, the group ring $\mathbb{C}\Gamma$ contains $\mathbb{C}A$ as a central subring. Furthermore, as A is torsion free, the non-zero elements of $\mathbb{C}A$ do not divide zero in $\mathbb{C}\Gamma$. Let $S = \mathbb{C}A - \{0\}$. We denote by $S^{-1}\mathbb{C}\Gamma$ the (central) localization of $\mathbb{C}\Gamma$ in which the elements of S are inverted.

Theorem 0.1. *If G is a finitely generated group, the global dimension of the ring $S^{-1}\mathbb{C}\Gamma$ is independent of the presentation, i.e. is an invariant of G.*

We denote this number by $\xi(G)$. It is always bounded above by $\mathrm{cd}_{\mathbb{C}}(G)$, see Section 2. On the other hand, if G is not a torsion group, $\xi(G) \geq 1$.

Theorem 0.2. *If G is a finitely generated free abelian group, then $\xi(G) = 1$.*

Thus it is possible that $\xi(G) < \mathrm{cd}_{\mathbb{C}}(G)$.

Theorem 0.2 is trivial if $\mathrm{rank}(G) = 1$ and was proved by Shamsuddin [9] if $\mathrm{rank}(G) = 2$. Our proof of the general case follows Roos' method [5] where he showed that the global dimension of the Weyl algebra A_n is n. This is done in Section 4.

On the other hand in Section 3 we prove

Theorem 0.3. *If $G = G_1 \times \cdots \times G_n$ where the G_i's are finitely generated free non-abelian groups, then $\xi(G) = \mathrm{cd}(G) = n$.*

Thus the inequality $1 \leq \xi(G) \leq \mathrm{cd}_{\mathbb{C}}(G)$ is 'best possible' in general.

To prove Theorem 0.1 we observe that $S^{-1}\mathbb{C}\Gamma$ can be viewed as a crossed product $K^{\alpha}G$ where $K = S^{-1}(\mathbb{C}A)$ and α is obtained from the extension $1 \to A \to \Gamma \to G \to 1$. We show that if one changes a free presentation $F \to G \to 1$ by adding to F a new generator which goes to 1 in G, then $1 \to A \to \Gamma \to G \to 1$ becomes

$$1 \to A \times \mathbb{Z} \to \Gamma \times \mathbb{Z} \to G \to 1$$

so that the field K becomes $K(x)$ where x is an indeterminate. Thus one is reduced to proving that $\mathrm{gl.dim}(K^{\alpha}G) = \mathrm{gl.dim}(K(x)^{\alpha}G)$ and this is done in Section 2.

Let G be a group. An extension of G, in this paper, is always assumed to have an abelian kernel. Let $\hat{\beta}$ be one:

$$\hat{\beta}: 1 \to A \to \Gamma \to G \to 1 .$$

The action of G on A (which is trivial in the central case) plus the class of extensions in $H^2(G, A)$, which we denote by β, completely determine it. If H is a subgroup of G, we say that the extension splits over H if there exists a morphism $\varphi: H \to \Gamma$ with $\pi \circ \varphi = \mathrm{id}_H$. This is equivalent to saying that $\mathrm{res}_H^G(\beta) = 0$.

Let now $F \to G$ be a presentation and

$$\hat{\alpha}: 1 \to A \to \Gamma \to G \to 1$$

its associated torsion free central extension. In Section 1 we prove

Proposition 0.4. *The set of subgroups of G over which α splits, $S_\alpha(G)$, is independent of the presentation, i.e. it is an invariant of G.*

Thus we can denote this set by $S(G)$. All our estimates from below of $\xi(G)$ are based on the principle that

$$\xi(G) \geq \sup\{\mathrm{cd}_{\mathbb{C}}(H) \colon H \in S(G)\} .$$

1. Tietze transformations and a generalization

A based presentation for a group G is a presentation $\pi: F \to G$ and a set of elements of $R = \ker(\pi)$, called 'defining relations', which generate R as a normal subgroup. A based presentation is finite if F is finitely generated and the number of defining relations is finite. Two based presentations $(F, \{r_i\}_{i \in I})$, $(F', \{r'_j\}_{j \in J})$ are isomorphic if there is an isomorphism $F \to F'$ sending $r_i \to r'_{\lambda(i)}$ where $\lambda: I \to J$ is a bijection. A Tietze transformation for $(F, \{r_i\})$ is either

(T1) add a relation $r \in R$ to the set of defining relations or

(T2) add a new generator x to F and a defining relation $x = w$ where $w \in F$.

Tietze's theorem, see [4, p.89], is that given two finite based presentations of the same group, there are finitely many Tietze transformations which render them isomorphic. We are interested in the effect of these transformations on $F/[F, R]$. It is obvious that (T1) does not change it.

Lemma 1.1. *Let $(F, \{r_i\})$ be a based presentation of G and $(F', \{r_i\} \cup \{x^{-1}w\})$ a presentation obtained from it by a type (T2) transformation. Let $\Gamma = F/[F, R]$, $\Gamma' = F'/[F', R']$. Then $\varphi: \Gamma' \xrightarrow{\simeq} \Gamma \times \mathbb{Z}$ by an isomorphism φ that commutes with the projection onto G, i.e. if π is the projection $\Gamma \to G$, π' the projection $\Gamma' \to G$*

and $p: \Gamma \times \mathbb{Z} \to \Gamma$ *projection on the first factor, then* $\pi' = \pi \circ p \circ \varphi$. *A similar result holds after dividing out the torsion in* $\ker(\pi)$ *and* $\ker(\pi')$.

Note that the lemma implies that Torsion (Γ) is an invariant of G.

Proof. Let $y = x^{-1}w$. There is a retraction $h: F' \to F$ which is the identity on F and sends y to 1. Note that both the inclusion $F \hookrightarrow F'$ and h are maps 'over G'. To show that the inclusion induces an injection of Γ onto Γ' we need to show that $[F', R'] \cap F = [F, R]$. Let $z \in [F', R'] \cap F$; as $z \in F$, $h(z) = z$. On the other hand $z \in [F', R']$ implies that $h(z) \in [F, R]$, so $z \in [F, R]$. Since F and y generate F', their images Γ and $\bar{y}$ generate Γ' and as $y \in R'$, $\bar{y}$ is central in Γ'. It remains to show that, identifying Γ with its image in Γ', no non-zero power of $\bar{y}$ lies in Γ. It is clear that there exists a morphism $\psi: F' \to \mathbb{Z} \times G$ uniquely defined by

(i) on F it is the projection to G,

(ii) it sends y to a generator of $\mathbb{Z}$, i.e. to $(1,1) \in \mathbb{Z} \times G$.

We claim $[F', R'] \subseteq \ker(\psi)$. If $u \in F'$, $w \in R'$, then w is a product, in some order, of conjugates of y and elements of R; thus $\psi(w) = (m, 1)$ for some integer m. Clearly this commutes with $\psi(u)$, so $\psi([u, w]) = [\psi(u), \psi(w)] = 1$. Thus ψ defines a morphism $\bar{\psi}: \Gamma' \to \mathbb{Z} \times G$. If $\bar{y}^e \in \Gamma$, then $\bar{\psi}(\bar{y}^e)$ lies in $\bar{\psi}(\Gamma) = \{0\} \times G$; since $\bar{\psi}(\bar{y}^e) = (e, 1)$, we see that $e = 0$, which completes the proof. □

It is easily checked that the isomorphism $\Gamma' \simeq \Gamma \times \mathbb{Z}$ just described takes the torsion of $\ker(\pi)$ onto the torsion of $\ker(\pi')$; this justifies the last statement in the lemma.

Strebel has suggested the following extension of Lemma 1.1. Let $\pi: F \to G \to 1$ be a free presentation of G, L a free group and $\nu: L \to G$ a homomorphism. Then there is a unique morphism $(\pi, \nu): F * L \to G$ which restricts to π on F and to ν on L. We want to compare the central extensions associated with π and (π, ν).

Lemma 1.2 (Strebel). *With the above notation let* $R = \ker(\pi)$, $\tilde{R} = \ker(\pi, \nu)$, *then* $F * L/[F * L, \tilde{R}] \simeq F/[F, R] \times \{\text{free abelian group}\}$, *the isomorphism being over* G. *More precisely, if* $1 \to A \to \Gamma \to G \to 1$ *is the central extension associated with* π, *then the central extension associated with* (π, ν) *is*

$$1 \to A \times \mathbb{Z}^d \to \Gamma \times \mathbb{Z}^d \to G \to 1$$

where $d = \text{rank}(L)$. □

The proof of Lemma 1.2 is much like that of Lemma 1.1 and is omitted.

Proof of Proposition 0.4. Given two presentations we must show that they determine the same set $S(G)$. Now, using Strebel's idea, any two can be compared with a third: their product; i.e. if $\pi : F \to G$, $\tilde{\pi} : \tilde{F} \to G$ are presentations, then $(\pi, \tilde{\pi}) : F * \tilde{F} \to G$ can be compared to both. If the associated central extensions are

$$\hat{\alpha} : 1 \to A \to \Gamma \to G \to 1\,, \qquad \hat{\tilde{\alpha}} : 1 \to \tilde{A} \to \tilde{\Gamma} \to G \to 1$$

and

$$\hat{\beta} : 1 \to B \to \Gamma_1 \to G \to 1$$

then by Lemma 1.2 $S_\alpha(G) = S_\beta(G)$ and $S_{\tilde{\alpha}}(G) = S_\beta(G)$. So $S_\alpha(G) = S_{\tilde{\alpha}}(G)$. $\square$

2. Crossed products

Let K be a field and G a group acting on it via a homomorphism $t : G \to \mathrm{Aut}(K)$. This endows K^* with a G-module structure, and if $\alpha \in H^2(G, K^*)$, one constructs the crossed product $K_t^\alpha(G)$ as usual (see [1]). It is the direct sum $\coprod_{\sigma \in G} Ku_\sigma$ where multiplication is defined by the rule $xu_\sigma \cdot yu_\tau = x\sigma(y)f(\sigma, \tau)u_{\sigma\tau}$ $(x, y \in K;\ \sigma, \tau \in G)$ where f is a cocycle representing α. In this paper we mostly need 'central' crossed products, i.e. those with trivial action $(\mathrm{Im}(t) = \{1\})$ and we denote them by $K^\alpha G$, dropping the trivial t. Similarly a crossed product in which α is trivial (i.e. $=0$) is denoted by $K_t G$.

A major source of crossed products is the localization of group rings. Let

$$\hat{\alpha} : 1 \to A \to \Gamma \to G \to 1$$

be an extension of G with A torsion free (and, as usual, abelian). Conjugation in Γ defines an action of G on A (trivial in the central case) and the extension determines a class $\alpha \in H^2(G, A)$. If k is a field and $K = k(A)$, the field of fractions of the group ring kA, then G acts on K via its action on A and the inclusion $\mathrm{inc} : A \hookrightarrow K^*$, which by the definition of the action of G on K is G-linear, induces a cohomology class $\mathrm{inc}_*(\alpha) \in H^2(G, K^*)$. By abuse of notation we drop the inc_* and call this class α, too. Now let $S = kA - \{0\} \subset k\Gamma$. In [6] it is shown that the localization $S^{-1}k\Gamma$ exists (but this is no problem in the central case).

Proposition 2.1. *With the above notation, $S^{-1}k\Gamma$ is isomorphic as a ring (and even as a K^G algebra, where K^G is the fixed field) with $K_t^\alpha G$.* $\square$

It will be useful, notationally, to describe an isomorphism. Choose representatives v_σ $(\sigma \in G)$ for the elements of G in Γ. This determines a cocycle f (with values in A). Use this f to construct $K_t^\alpha G$. Then map $S^{-1}k\Gamma$ to $K_t^\alpha G$ by sending $S^{-1}kA$ to K (by the 'identity') and v_σ to u_σ.

If H is a subgroup of G and $t' = t\,|\,H$, $\alpha' = \mathrm{res}_H^G(\alpha)$, there is a natural way to view $K_{t'}^{\alpha'} H$ as a subring of $K_t^\alpha G$ and by [1, Proposition 4.1] we have

Proposition 2.2. $\mathrm{gl.dim}(K_{t'}^{\alpha'}H) \leq \mathrm{gl.dim}(K_t^{\alpha}G)$, *if both are finite.* $\square$

As an example suppose

$$\hat{\alpha}: 1 \to A \to \Gamma \to G \to 1$$

is an extension and H a subgroup of G over which the extension splits. Then $\mathrm{res}_H^G(\alpha) = 0$. If Γ_H is the inverse image of H, then the proposition says

$$\mathrm{gl.dim}(S^{-1}k\Gamma_H) \leq \mathrm{gl.dim}(S^{-1}k\Gamma)\,.$$

Now we identify $S^{-1}k\Gamma_H$, as above, with K_tH. In [1, 4.6] it is shown that $\mathrm{gl.dim}(K_tH) = \mathrm{cd}_k H$. Thus

Proposition 2.3. If $1 \to A \to \Gamma \to G \to 1$ *is an extension, then* $\mathrm{gl.dim}(S^{-1}k\Gamma) \geq \sup\{\mathrm{cd}_k(H)$: *the extension splits over* $H\}$. $\square$

We now turn to field extensions. Suppose $K \overset{\iota}{\hookrightarrow} F$ are fields on which G acts compatibly, i.e. the inclusion is a G map. We denote both maps $G \to \mathrm{Aut}(K)$, $G \to \mathrm{Aut}(F)$ by t. If $\alpha \in H^2(G, K^*)$, let $\beta = \iota_*(\alpha)$. It is clear how $K_t^{\alpha}G$ imbeds in $F_t^{\beta}G$. In [1, 4.4] we proved

Proposition 2.4. *With the above notation* $\mathrm{gl.dim}(K_t^{\alpha}G) \leq \mathrm{gl.dim}(F_t^{\beta}G)$, *if both are finite.* $\square$

As an important example for the above, suppose $F = K(x)$, a field of rational functions in the variable x, and the action of G on K is extended to act trivially on x. Then

Problem 2.5. Is it always true in this case that $\mathrm{gl.dim}(K_t^{\alpha}G) = \mathrm{gl.dim}(F_t^{\beta}G)$?

It is because I cannot prove this that the construction of ξ is done over $\mathbb{C}$ and not over an arbitrary field k. The result we prove instead is rather more restricted. As always round brackets, such as in $\mathbb{C}(A)$, denote 'field of fractions'.

Proposition 2.6. *Let* $\hat{\alpha}: 1 \to A \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1$ *be a group extension where* A *is torsion free. Let* $\hat{\beta}: 1 \to A \times B \xrightarrow{\iota \times 1} \Gamma \times B \xrightarrow{\pi'} G \to 1$ *where* B *is a finitely generated free abelian group and* π' *is defined by* $\pi'(\gamma, s) = \pi(\gamma)$. *Let* $K = \mathbb{C}(A)$, $F = \mathbb{C}(A \times B)$, $\Lambda = K_t^{\alpha}G$ $(= S^{-1}\mathbb{C}\Gamma$ *with* $S = \mathbb{C}A - \{0\})$ *and* $\tilde{\Lambda} = F_t^{\beta}G$ $(= S^{-1}\mathbb{C}[A \times B]$ *with* $S = \mathbb{C}[A \times B] - \{0\})$. *Then* $\mathrm{gl.dim}(\Lambda) = \mathrm{gl.dim}(\tilde{\Lambda})$.

Proof. By Proposition 2.2 $\mathrm{gl.dim}\,\Lambda \leq \mathrm{gl.dim}\,\tilde{\Lambda}$. If $x_1, \ldots, x_r$ is a basis of B, we identify x_i with $(1, x_i)$ in $\Gamma \times B$. Clearly $F = K(x_1, \ldots, x_r) = \mathbb{C}(x_i, \ldots, x_r)(A)$. There is an embedding of $\mathbb{C}(x_1, \ldots, x_r)$ in $\mathbb{C}$; let its image be E. Then

$\tilde{\Lambda} = E(A)_t^\alpha G$ and by Proposition 2.2 again

$$\mathrm{gl.dim}(\tilde{\Lambda}) \leq \mathrm{gl.dim}(\mathbb{C}(A)_t^\alpha G) = \mathrm{gl.dim}(\Lambda) .$$

This completes the proof. $\square$

We can now prove the main result.

Proof of Theorem 0.1. Let $\pi : F \to G \to 1$ and $\pi' : F' \to G \to 1$ be two presentations of G where F, F' are finitely generated. Let $1 \to A \to \Gamma \to G \to 1$ and $1 \to A' \to \Gamma' \to G \to 1$ be the associated central extensions and Λ, Λ' the associated rings (as in Proposition 2.6). Let $\tilde{F} = F * F'$ and $\tilde{\pi} : \tilde{F} \to G$ the map restricting to π on F and to π' on F'. Let $1 \to \tilde{A} \to \tilde{\Gamma} \to G \to 1$ and $\tilde{\Lambda}$ be the associated central extension and ring. According to Lemma 1.2 and Proposition 2.6, $\mathrm{gl.dim}(\Lambda) = \mathrm{gl.dim}(\tilde{\Lambda})$ and $\mathrm{gl.dim}(\Lambda') = \mathrm{gl.dim}(\tilde{\Lambda})$, and the proof is complete. $\square$

We observe that in [1, 3.4] it is proved that $\mathrm{gl.dim}(K_t^\alpha G) \leq \mathrm{cd}_K(G)$. Since $\xi(G)$ is defined to be $\mathrm{gl.dim}(S^{-1}\mathbb{C}\Gamma)$, we see from the above discussion that

Proposition 2.7. $\xi(G) \leq \mathrm{cd}_{\mathbb{C}} G$. $\square$

(Note that $\mathrm{cd}_K G$ only depends on the characteristic of K.)

This shows that ξ vanishes on the class of finite groups. The question now arises whether ξ can change upon passage to subgroups of finite index. J. Moody has observed that, remarkably, this can happen.

Example 2.8. (due to Moody). Consider $G = \langle a, b \colon aba^{-1} = b^{-1} \rangle = \langle b \rangle \rtimes \langle a \rangle = F/R$, and the subgroup $H = \langle a^2, b \rangle = \langle b \rangle \times \langle a^2 \rangle$ of index 2. Here F is the free group on a, b and R its normal subgroup generated normally by $aba^{-1}b$. We have

$$\Gamma = F/[F, R] = \langle s, t \colon [tsts^{-1}, s], [tsts^{-1}, t] \rangle .$$

Let $x = tsts$, $y = t(tsts^{-1})^{-1} = tst^{-1}s^{-1}t^{-1}$ (in Γ). Under $\Gamma \to G$, where s maps to a and t to b, the elements x, y map to a^2, b respectively. We compute the commutator of x and y,

$$\begin{aligned}[x, y] &= (tsts)(tst^{-1}s^{-1}t^{-1})(tsts)^{-1} \cdot (tst^{-1}s^{-1}t^{-1})^{-1} \\ &= ts(tst)s(tst)^{-1}s^{-1}t^{-1}(s^{-1}t^{-1}ts)ts^{-1}t^{-1} .\end{aligned}$$

From $[tsts^{-1}, s] = 1$ we see that s commutes with tst, so

$$[x, y] = ts \cdot s \cdot s^{-1}t^{-1} \cdot ts^{-1}t^{-1} = 1 .$$

Thus, by sending a^2 to x and b to y, we see that $\Gamma \to G$ splits over H, so $H \in S(G)$ and $\xi(G) \geq \mathrm{cd}(H) = 2$. As $\mathrm{cd}(G) = 2$, we see that $\xi(G) = 2$. On the other hand $\xi(H) = 1$ by Proposition 0.4.

3. Products of free groups

We shall use the following easily checked

Lemma 3.1 (Identity). $x[y, z]x^{-1} = (xy)z(xy)^{-1} \cdot z^{-1} \cdot zxz^{-1}x^{-1} = [xy, z][z, x]$. $\square$

This can also be rewritten as

$$[xy, z] = [x, [y, z]][y, z][x, z].$$

Now suppose F_1, F_2 are finitely generated non-commutative free groups. Let $G = F_1 \times F_2$ and $F = F_1 * F_2$ (their 'sum'). An obvious presentation for G is

$$(*) \qquad 1 \to R \to F_1 * F_2 \to F_1 \times F_2 \to 1$$

and one knows [8, p.6] that R is the free group on the commutators $[y, z]$, $y \in F_1 - \{1\}$, $z \in F_2 - \{1\}$. Now let $A = R/[F, R]$, $\Gamma = F/[F, R]$, so that

$$(**) \qquad 1 \to A \to \Gamma \xrightarrow{\pi} G \to 1$$

is the associated central extension (and, in fact, A is torsion free). There is an obvious splitting of $(*)$ over F_1 (sending $x \in F_1$ to $x \in F_1 * F_2$) and similarly over F_2. These induce splittings ρ_1, ρ_2 of $(**)$ over F_1, F_2 respectively. If $x, y \in F_1$, $z \in F_2$, then, in G, $[xy, z] = [y, z] = 1$, while in Γ since $[\rho_1(y), \rho_2(z)] \in A$, which is central,

$$[\rho_1(x), [\rho_1(y), \rho_2(z)]] = 1 .$$

By Lemma 3.1 it follows that $[\rho_1(xy), \rho_2(z)] = [\rho_1(x), \rho_2(z)] \cdot [\rho_1(y), \rho_2(z)]$.

Fixing z we see that $x \mapsto [\rho_1(x), \rho_2(z)]$ is a homomorphism of F_1 into the commutative group A. It is, therefore, trivial on $[F_1, F_1]$. It follows that the subgroups $\rho_1([F_1, F_1])$ and $\rho_2(F_2)$ commute elementwise and Γ contains a subgroup, their product, which is mapped by π isomorphically onto $[F_1, F_1] \times F_2$. In other words, $(**)$ splits over $[F_1, F_1] \times F_2$. Since $[F_1, F_1] \times F_2$ and $F_1 \times F_2$ have cohomological dimension 2 (see [7, p.87]), we see by Propositions 2.3 and 2.4 that we proved the case $n = 2$ of the following proposition:

Proposition 3.2. *If $F_1, \ldots, F_n$ are free groups, then every central extension of $F_1 \times \cdots \times F_n$ splits over $[F_1, F_1] \times \cdots \times [F_{n-1}, F_{n-1}] \times F_n$. In particular if the F_i's are finitely generated and non-commutative, then $\xi(\Pi_{i=1}^n F_i) = n$ $(= cd(\Pi_{i=1}^n F_i))$.* $\square$

The proof of this fact is a straightforward extension of the argument above. But note that we gave an explicit splitting of $(**)$ over F_1 whereas in the proof of Proposition 3.2 one can only argue (since the extension is not given) that such a splitting exists because F_1 is free.

A class of groups which is closely related to the class of free groups (and includes it) is the class of 1-relator groups. For example $\mathbb{Z} \times \mathbb{Z}$ is a 1-relator group. We know that for a 1-relator group G, $\xi(G)$ can be either 1 or 2 (unless G is finite). That both possibilities can occur was observed by several people: $\xi(\mathbb{Z} \times \mathbb{Z}) = 1$; while if $G = \mathbb{Z} \times \mathbb{Z}$, where the action of the second factor on the first is non-trivial, then $\xi(G) = 2$, as shown above (Example 2.8).

4. Free abelian groups

Let G be a free abelian group on n generators. We shall prove

Theorem 4.1. $\xi(G) = 1$.

The result is trivial if $n = 1$ and is known if $n = 2$ [9]. We follow Roos' proof [5] that the Weyl algebra A_n has global dimension n; but the non-commutativity here is more pronounced making the present proof more difficult.

Since for a Noetherian ring the global dimension (defined by the vanishing of Ext's) equals the weak (or Tor) global dimension (defined by flat resolutions and the vanishing of Tor's) and since all rings in this section are Noetherian, we work with Tor functors.

Before embarking on the proof we recall two constructions.

(1) If P is a ring, a 'twisted Laurent extension' of P is a ring $P_\sigma[x, x^{-1}]$ where x is a variable, σ an automorphism of P and for $a \in P$, $xa = \sigma(a)x$. It is clear that if P is a Noetherian domain, so is $P_\sigma[x, x^{-1}]$. For the applications below we note that if a group H is a semi-direct product $H' \rtimes \mathbb{Z}$, then the group ring kH is a twisted Laurent extension of $P = kH'$.

(2) If P is a ring and W a multiplicatively closed subset consisting of non-zero-divisors, then the (left) localization $W^{-1}P$ is a ring containing P such that the elements of W are invertible in it and every element in it can be written as $a^{-1}b$ for some $a \in W$, $b \in P$ (see [2]). Necessary and sufficient for $W^{-1}P$ to exist are the so-called Öre conditions. If they hold and M is a P module, then also $W^{-1}M$ can be defined and, as for commutative localizations, there is a natural isomorphism of $W^{-1}P$ modules $W^{-1}P \otimes_P M \xrightarrow{\sim} W^{-1}M$. It follows that if an element $m \in M$ is such that $1 \otimes m = 0$ in $W^{-1}P \otimes_P M$, then there is a $w \in W$ such that $wm = 0$.

Note also that $W^{-1}P$ is a flat right module and since left and right localizations are equal (when both exist) it is left flat too.

Returning now to our free abelian G let $\tau_1, \ldots, \tau_n$ be a basis for G and let F be a free group on n generators $a_1, \ldots, a_n$. F maps onto G by a morphism sending a_i to τ_i. The kernel of this is $R = [F, F]$, generated normally by the commutators $[a_i, a_j]$, $1 \le i < j \le n$. It is well known that $A = R/[F, R]$ is a free abelian group with the residue classes $\overline{[a_i, a_j]}$ as basis. (In fact $R/[F, R] = H_2(G, \mathbb{Z})$.) Let $\Gamma = F/[F, R]$. One can think of the extension

$$(*) \qquad 1 \to A \to \Gamma \to G \to 1$$

as the 'most general' central extension of G in the following sense. Let $\delta_1, \ldots, \delta_n$ be a basis of G and $f : G \times G \to \Lambda^2(G)$ the bilinear map satisfying $f(\delta_i, \delta_j) = \delta_i \wedge \delta_j$ if $i \le j$ and $=0$ if $i > j$. Then f is a cocycle and the central extension of G by $\Lambda^2(G)$ with cocycle f is isomorphic to $(*)$, i.e. there is an isomorphism $A \to \Lambda^2(G)$ which takes the cohomology class of $(*)$ to the class of F.

We denote $\bar{a}_i$ by u_i and the commutator $[u_i, u_j]$ by b_{ij} (it is the 'bond' of i and j). Let $S = \mathbb{C}A - \{0\} \subset \mathbb{C}\Gamma$ and $B = S^{-1}\mathbb{C}\Gamma$. It is isomorphic to $K^\alpha G$ where $K = \mathbb{C}(A) = S^{-1}\mathbb{C}A$ and α comes from $(*)$ (or, equivalently, $\alpha = [f]$).

We will need later the following well known

Lemma 4.2. *B is a simple ring.*

Proof. Let $I \ne \{0\}$ be a 2-sided ideal. We are to prove $I = (1)$. Every element of B is, uniquely, a finite sum of 'monomials' $u^{(i)} = u_1^{i_1} \ldots u_n^{i_n}$ with non-zero coefficients from K. Call $l(\cdot)$ the number of such summands and let $0 \ne g \in I$ with $l(g)$ minimal. Modifying g, if necessary, we can assume 1 appears in it, i.e. $g = 1 +$ (other monomials). If $g = 1$, fine. If not, we get a contradiction by showing it is not minimal as follows. Say $u_1^{i_1} \ldots, u_n^{i_n}$ appears in g with $i_1 \ne 0$. Then one sees that, since b_{ij} are multiplicatively independent, $g - u_2^{-1} g u_2 \ne 0$ and is shorter than g (the 1 has fallen). This completes the proof. $\square$

Let $G_i = \{\tau_1, \ldots, \tau_{n-i}\rangle$, $i = 0, \ldots, n$ so that $G_0 = G, \ldots, G_n = \{1\}$, and $G_i = G_{i+1} \times \langle \tau_{n-i} \rangle$. Let $\Gamma_i = \pi^{-1}(G_i)$. It is an extension

$$1 \to A \to \Gamma_i \to G_i \to 1$$

where A is $\Lambda^2(G_i)$ *plus* some 'irrelevant' copies of $\mathbb{Z}$ corresponding to commutators involving u_{n-j} with $j < i$. By Proposition 2.6, gl.dim$(S^{-1}\mathbb{C}\Gamma_i) = \xi(G_i)$. We also note that a similar proof to that of Lemma 4.2 shows that $S^{-1}\mathbb{C}\Gamma_i$ is a simple ring for $i = 1, \ldots, n$.

We will prove the following theorem, which includes Theorem 4.1.

Theorem 4.3. *For $j = 1, \ldots, n$ let $S_j = \mathbb{C}\Gamma_j - \{0\}$ (so that $S = S_n$). Then*
(i) *The elements of S_j do not divide zero in $\mathbb{C}\Gamma$;*
(ii) *The localization $S_j^{-1}\mathbb{C}\Gamma$ exists;*
(iii) *For $j \geq 2$, if M is a module over $S_j^{-1}\mathbb{C}\Gamma_{j-2}$ which is finite dimensional over the division ring $S_j^{-1}\mathbb{C}\Gamma_j$, then $M = \{0\}$;*
(iv) $\mathrm{gl.dim}(S_j^{-1}\mathbb{C}\Gamma) = 1$ *for $j = 1, \ldots, n$.*

Proof. (i) Since $\mathbb{C}\Gamma$ is constructed from $\mathbb{C}A$ by successive twisted Laurent extensions, it is a domain. Thus (i) is clear.

(ii) We note first that for each j the ring $\mathbb{C}\Gamma_j$ has a total ring of fractions, $\mathbb{C}(\Gamma_j)$, which is a skew field. This can be proved directly (e.g. by induction: start with $\mathbb{C}A$ and work 'up') or by referring to Goldie's theorem (cf. [2]). Now let $1 < j$. We show that if $S_j^{-1}\mathbb{C}\Gamma_l$ exists, then $S_j^{-1}\mathbb{C}\Gamma_{l-1}$ exists. Starting with $l = j$ and going down to $l = 0$ will prove (ii). Now the group Γ_{l-1} (supposing $l > 0$) is a semi-direct product $\Gamma_l \rtimes \mathbb{Z}$. Conjugation by a generator of the $\mathbb{Z}$ factor gives an automophism, σ, of Γ_l. This automorphism can be extended to an automorphism of $\mathbb{C}\Gamma_l$ and, since S_j is preserved by it, also to the localization $P = S_j^{-1}\mathbb{C}\Gamma_l$. The twisted Laurent extension $P_\sigma[x, x^{-1}]$ contains $\mathbb{C}\Gamma_{l-1}$, the elements of S_j are invertible in it and clearly every element in it has the form $s^{-1}r$ where $s \in S$, $r \in \mathbb{C}\Gamma_{l-1}$. This proves (ii).

(iii) It would suffice to treat the case $j = 2$, since this is the most difficult case and the others can be reduced to it. Thus we have a skew field $E = S_2^{-1}\mathbb{C}\Gamma_2$ and a ring $Y = S_2^{-1}\mathbb{C}\Gamma = E[u_{n-1}^{\pm 1}, u_n^{\pm 1}]$ where u_{n-1}, u_n act on E by conjugation and $u_{n-1}u_n = \epsilon u_n u_{n-1}$ where ϵ $(= b_{n-1,n})$ is in the center of E. Suppose M is a Y-module (on which 1 acts as the identity), finite dimensional over E. We must prove that $M = 0$. If E is central in Y (which holds iff $\mathrm{rank(G)} \leq 2$), this is easy and done as follows. Multiplication by elements of Y defines E-linear endomorphisms of M and, hence, a ring homomorphism $Y \to \mathrm{End}_E(M)$. This ring map cannot be injective since $\mathrm{End}_E(M)$ is finite dimensional over E and Y is not. As Y is a simple ring, this map must be 0 which implies that $M = 0$. Now assume $n > 2$ and the result proved for a smaller number of 'variables', i.e. for G of rank less than n. We shall 'specialize' one of the variables, say u_1, thereby reducing n to $n - 1$. But this will require some effort. Recall that E is given as follows. First there is the field $K = \mathbb{C}(b_{ij})$, $1 \leq i < j \leq n$, $b_{ji} = b_{ij}^{-1}$. Then $E = K(u_1, \ldots, u_{n-2})$ where $u_i u_j u_i^{-1} u_j^{-1} = b_{ij}$. Let $e_1, \ldots, e_r$ be a basis for M over the skew field E. For $i = 1, \ldots, r$ there exist $p_{ij} \in E$ such that

$$u_{n-1}e_i = \sum_{j=1}^{r} p_{ij} \cdot e_j \,.$$

Similarly $u_{n-1}^{-1}e_i$ is expressible as $\sum \bar{p}_{ij}e_j$ and $u_n e_i = \sum q_{ij}e_j$, $u_n^{-1}e_i = \sum \bar{q}_{ij}e_j$. Each p_{ij}, $\bar{p}_{ij}$, q_{ij}, $\bar{q}_{ij}$ is a rational function of the form $f_{ij}^{-1}g_{ij}$ where f_{ij}, g_{ij} lie in the ring generated over K by $u_1^{\pm1}, \ldots, u_{n-2}^{\pm1}$. Each denominator appearing in p_{ij}, $\bar{p}_{ij}$, q_{ij} or $\bar{q}_{ij}$ is a polynomial of type

$$(*) \qquad \sum c_\alpha M_\alpha \quad \text{(finite sum)}$$

where M_α is a monomial $u_2^{\alpha_2} \ldots u_{n-2}^{\alpha_{n-2}}$ in $u_2^{\pm1}, \ldots, u_{n-2}^{\pm1}$ (if $n \leq 3$, these M_α equal 1) and $c_\alpha \in K[u_1, u_1^{-1}]$. We observe that if an expression of type $(*)$ is conjugated by some u_j, it is transformed into a similar expression with the following changes:

(CI) c_α gets multiplied by a monomial in the b_{ij}'s and

(CII) c_α itself gets changed as follows. If c_α was equal to $\Sigma\, c_{\alpha,\beta} u_1^\beta$ ($c_{\alpha,\beta} \in K$), then it gets changed to $\Sigma\, c'_{\alpha,\beta} u_1^\beta$ where each $c'_{\alpha,\beta}$ differs from $c_{\alpha,\beta}$ by some power of b_{1j}.

This will now be used.

Claim. There exists an integer m satisfying the following. Let w be an mth root of b_{23}. Then for some choice of mth roots of unity (not necessarily primitive) of order m $\zeta_2, \ldots, \zeta_n$ the substitution

$$c_\alpha(b_{ij}, u_1) \mapsto c_\alpha(\zeta_2, \ldots \zeta_n, b_{23}, \ldots, b_{n-1,n}, w)$$

is permissible and not 0 for all c_α of type $(*)$ (i.e. appearing in denominators of some p_{ij}, $\bar{p}_{ij}$, q_{ij} or $\bar{q}_{ij}$) and their conjugates by $u_1, \ldots, u_n$.

Proof. Each c_α is a polynomial in $K[u_1, u_1^{-1}]$ and thus a finite sum $\Sigma_\beta c_{\alpha,\beta} u_1^\beta$ where $c_{\alpha,\beta} \in K$. Let C be the product of all these $c_{\alpha,\beta}$ which appear in p_{ij}'s, $\bar{p}_{ij}$'s, q_{ij}'s or $\bar{q}_{ij}$'s. It is a rational function in the variables b_{ij} and as such there certainly exist roots of unity $\zeta_2, \ldots, \zeta_n$ such that plugging ζ_j for $b_{i,j}$ $(j = 2, \ldots, n)$ is allowed and gives a non-zero result. Now choose m large enough so that the ζ_j are all mth roots of 1 and the expressions $c_\alpha(\zeta_2, \ldots, \zeta_n, b_{23}, \ldots, b_{n-1,n}, w)$ are all non-zero. By the remark before the claim also the conjugates of the c_α's will not vanish. □

Let F be the field generated over $\mathbb{C}$ by the variables b_{ij} where $2 \leq i < j \leq n$ and let Φ be the ring generated over F by the denominators f_{ij}^{-1} (appearing in the elements p_{ij}, $\bar{p}_{ij}$, q_{ij}, $\bar{q}_{ij}$) and their conjugates by the u_j's, and by the variables $b_{1,j}^{\pm1}$ $(j = 2, \ldots, n)$ and $u_1^{\pm1}, \ldots, u_{n-2}^{\pm1}$. Let Φ' be the ring generated by Φ and $u_{n-1}^{\pm1}$, $u_n^{\pm1}$.

We will need a certain intermediate crossed product with non-trivial (but not faithful) action. Let $L = F(w)$; the free abelian group G' generated by $u_2^{\pm1}, \ldots, u_n^{\pm1}$ acts on L by acting trivially on F and by the formula

$$u_j w u_j^{-1} = \zeta_j w\,.$$

Let t denote this action. Then $L_t^\beta G'$ is the ring generated over L by $u_2^{\pm 1}, \ldots, u_n^{\pm 1}$ (i.e. by G') and β signifies that

$$u_i u_j u_i^{-1} u_j^{-1} = b_{ij} \in F^*$$

if $2 \leq i, j \leq n$. We denote the total ring of fractions of $L_t^\beta G'$ by $L_t^\beta(G')$. Of course this is a skew field. Let $\bar{f}_{ij}^{-1}$ be the elements (in $L_t^\beta(G')$) obtained by substituting w for u_1 and ζ_j for $b_{i,j}$ $(j = 2, \ldots, n)$ in the f_{ij}^{-1} mentioned above. Let Ψ' be the subring of $L_t^\beta(G')$ generated, over L, by $u_2^{\pm 1}, \ldots, u_n^{\pm 1}$ and the $\bar{f}_{ij}^{-1}$ and their conjugates by the u_j's.

There is a surjection $\Phi' \to \Psi'$, over F, sending u_1 to w, u_j to u_j and $b_{1,j}$ to ζ_j for $j = 2, \ldots, n$. It is not hard to see that the images of the c_α's are well defined by these assumptions. The image of Φ under this map, called Ψ, is the subring of Ψ' generated over L by $u_2^{\pm 1}, \ldots, u_{n-2}^{\pm 1}$.

Let X be the Φ-module generated by $e_1, \ldots, e_r$. It is a free Φ-module of rank r. Also $u_{n-1}^{\pm 1}$ and $u_n^{\pm 1}$ act on X by the formulas

$$u_{n-1} \sum_i a_i e_i = \sum_i u_{n-1} \cdot a_i u_{n-1}^{-1} \Big(\sum_j p_{ij} e_j\Big),$$

$$u_n \sum_i a_i e_i = \sum_i u_n a_i u_n^{-1} \cdot \sum_j q_{ij} e_j .$$

In fact these formulas, together with the analogous formulas for the action of the inverses

$$u_{n-1}^{-1} \sum_i a_i e_i = \sum_i u_{n-1}^{-1} a_i u_{n-1} \Big(\sum_j \bar{p}_{ij} e_j\Big),$$

$$u_n^{-1} \sum_i a_i e_i = \sum_i u_n^{-1} a_i u_n \Big(\sum_j \bar{q}_{ij} e_j\Big)$$

define an action of the ring Φ' on X. Now let

$$\tilde{X} = \Phi \otimes_\Phi X .$$

It is clear that $\tilde{X}$ is a Ψ-module that is finitely generated and free of rank r. Also Ψ' acts on X in the same way Φ' acts on X, i.e. by similar formulas. We now invert the non-zero elements of Ψ. This transforms Ψ into a division ring Δ and Ψ' into a ring Δ' generated over Δ by the two variables u_{n-1} and u_n. $\tilde{X}$ is changed into $\Delta \otimes_\Psi X$, a Δ-space of dimension r which is also a Δ'-module. But Δ' is a ring much like Y above, only with one variable less, i.e. u_1 has disappeared. More precisely Δ' contains a subring, Y', which is like Y with u_1 thrown out and Δ' is a free Y-module (of rank $\dim_F L$). Y' is defined as follows. Let $F^\beta G'$ be the subring of $L_t^\beta G'$ generated over F by the variables $u_2^{\pm 1}, \ldots, u_n^{\pm 1}$. Y' is obtained by inverting the non-zero elements of $F^\beta[u_2^{\pm 1}, \ldots, u_{n-2}^{\pm 1}] \subset F^\beta G'$. Denote the division ring of fractions of $F^\beta[u_2^{\pm 1}, \ldots, u_{n-2}^{\pm 1}]$ by Δ_0. Then, by our inductive assumption, Y-modules that are finite dimensional over Δ_0 are 0. But $\Delta \otimes_\Psi \tilde{X}$ is just that: its dimension over Δ_0 is $r \cdot \dim_F(L)$. Thus r must be zero. This implies that M was zero in the first place, proving (iii).

(iv) We note first that $S_1^{-1}\mathbb{C}\Gamma$ is a twisted Laurent extension of the skew field $S_1^{-1}\mathbb{C}\Gamma_1$. Thus for $j = 1$, (iv) is well known (say by [3, lemma 23]). Now let $j > 1$ (but $j \leq n$) and assume that result for $j - 1$. We can view $\Lambda = S_j^{-1}\mathbb{C}\Gamma$ as a twisted Laurent extension of $S_j^{-1}\mathbb{C}\Gamma_1$. Since Γ_1/Γ_j is a free abelian group of rank $j - 1$, the inductive hypothesis (and Proposition 2.6) gives us that $S_j^{-1}\mathbb{C}\Gamma_1$ has global dimension ≤ 1. Thus, by [3] again, $\text{gl.dim}(S_j^{-1}\mathbb{C}\Gamma) \leq 2$. This will be useful below.

Denote the skew field $\mathbb{C}(\Gamma_j) = S_j^{-1}\mathbb{C}\Gamma_j$ by D (j is fixed for the rest of this proof). Let $U = S_j^{-1}\mathbb{C}\Gamma_{j-2}$. It is generated over D by $u_{n-j+1} = u$ and $u_{n-j+2} = v$ and their inverses. We claim that U is a simple ring. Now U is a localization of $S^{-1}\mathbb{C}\Gamma_{j-2}$ and a localization of a simple ring is clearly simple. Thus U is simple if $S^{-1}\mathbb{C}\Gamma_{j-2}$ is simple. The proof that $S^{-1}\mathbb{C}\Gamma_{j-2}$ is simple is identical to the proof of Lemma 4.2.

Let D_1 (resp. D_2) be the ring generated over D by u and u^{-1} (resp. by v and v^{-1}). Let $W_i = D_i - \{0\}$, $i = 1, 2$. The localizations $\Lambda_i = W_i^{-1}\Lambda$ exist (recall $\Lambda = S_j^{-1}\mathbb{C}\Gamma$). In fact $W_1^{-1}\Lambda$ is just $S_{j-1}^{-1}\mathbb{C}\Gamma$. By the induction hypothesis, Λ_1, Λ_2 both have global dimension 1. Let M be a Λ-module. To prove (iv) it will suffice to show that the Tor dimension of $M \leq 1$. The map

$$\mu : M \to \Lambda_1 \otimes_\Lambda M \oplus \Lambda_2 \otimes_\Lambda M$$

defined by $m \mapsto (1 \otimes m, 1 \otimes m)$ is Λ-linear.

Claim. μ is injective.

Proof. Let $\mu(m) = 0$. As $1 \otimes m = 0$ in $\Lambda_1 \otimes_\Lambda M$, there is a $w \in W$ such that $wm = 0$. Now w is a Laurent polynomial in u with coefficients in the skew field D. Multiplying by a suitable element au^e (if necessary) we can assume $w = a_0 + a_1 u + \cdots + u^n$ where $a_i \in D$, $n \geq 0$ and $a_0 \neq 0$. Let $V = D_1 m$; it is finite dimensional over D. Indeed let $V' = Dm + Dum + \cdots + Du^{n-1}m$, then, as $wm = 0$, we see that $u^n m \in V'$, hence $u^{n+1} \cdot m \in V'$ and so on. Dividing by u, we see similarly that $u^{-1}m \in V'$ etc., so $V = V'$. Since $\ker(\mu)$ is a submodule of M, each $u^r m$ (r an integer) satisfies $1 \otimes u^r m = 0$ in $\Lambda_2 \otimes_\Lambda M$. As above the vector spaces (over D) $D_2 u^r m$ are finite dimensional and since in U every element is expressible as a sum (with D coefficients) of monomials $v^{i_1}u^{i_2}$, we see that Um is finite dimensional over D. By (iii) above, $Um = 0$ so $m = 0$ and μ is injective. $\square$

As a result we have an exact sequence (of Λ-modules)

$$0 \to M \to \Lambda_1 \otimes_\Lambda M \oplus \Lambda_2 \otimes_\Lambda M \to Q \to 0\,.$$

Since Λ_1 is (left and right) flat over Λ, a flat Λ_1-module is flat over Λ. Thus a flat resolution over Λ_1 is also a resolution over Λ. As $\mathrm{gl.dim}(\Lambda_1) = 1$, $\Lambda_1 \otimes_\Lambda M$ has a flat resolution of length 1 over Λ_1; so its Tor dimension over Λ is ≤ 1. The same applies, of course, to $\Lambda_2 \otimes_\Lambda M$. The piece of the Tor's long exact sequence around $\mathrm{Tor}_2^\Lambda(*, M)$ is

$$\mathrm{Tor}_3^\Lambda(*, Q) \to \mathrm{Tor}_2^\Lambda(*, M) \to \mathrm{Tor}_2^\Lambda(*, \Lambda_1 \otimes_\Lambda M) + \mathrm{Tor} \cdot (*, \Lambda_2 \otimes_\Lambda M) = 0.$$

But $\mathrm{Tor}_3^\Lambda(*, Q) = 0$ since we observed above that $\mathrm{gl.dim}(\Lambda) \leq 2$. So $\mathrm{Tor}_2^\Lambda(*, M) = 0$ and the Tor dimension of $M \leq 1$. This completes the proof. □

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